Communications in Contemporary Mathematics
(2021) 2150055 (15 pages)

DOI: 10.1142/S0219199721500553

A special Calabi–Yau degeneration with trivial monodromy

Sławomir Cynk

Institute of Mathematics, Jagiellonian University
ul. Łojasiewicza 6,30-348 Kraków, Poland
slawomir.cynk@uj.edu.pl

Duco van Straten

Institut für Mathematik, FB 08-Physik
Mathematik und Informatik, Johannes Gutenberg-Universität
Staudingerweg 9, 55128 Mainz, Germany
straten@mathematik.uni-mainz.de

Received 23 December 2020
Accepted 18 March 2021
Published 21 May 2021

A well-known theorem of Kulikov, Persson and Pinkham states that a degeneration of a family of K3-surfaces with trivial monodromy can be completed to a smooth family. We give a simple example that an analogous statement does not hold for Calabi–Yau threefolds.

Keywords: Calabi–Yau threefolds; semi-stable reduction; monodromy; Kulikov-Persson-Pinkham theorem.

Mathematics Subject Classification 2020: 14D05, 14D06, 14D07, 14J32

1. Introduction

In the study of degenerations of $K3$ surfaces, the theorems of Kulikov [12] Persson and Pinkham [16] are fundamental and play a key role in the proof of the surjectivity of the period map for $K3$ surfaces. One important result concerns *degenerations of type I*: if $f : \mathcal{X} \to \Delta$ is a degeneration of $K3$ surfaces over the disc $\Delta := \{t \in \mathbb{C} \,|\, |t| < 1\}$, with monodromy on $H^2(X_t)$ of *finite order*, then after an appropriate base change and birational modification of the zero-fiber we obtain a family for which the fiber X_0 over 0 is *smooth*. For example, if $\mathcal{X} \to \Delta$ acquires only ADE-singularities at the zero-fiber, then the theorem is implied by the phenomenon of *simultaneous resolution after base change* for these singularities, discovered by Brieskorn [2], generalizing the A_1-case of Atiyah [1]. The fact that for ADE-singularities in dimension two resolving and deforming "are the same" has

*Corresponding author.

been recognized as a typical feature in hyperkähler geometry and recently a generalization of the above theorem to degenerations of higher dimensional hyperkähler manifolds was given in [11]. As remarked in that paper, it is known that this does not generalize to degenerations of higher dimensional *Calabi–Yau* varieties. The homological monodromy of an odd-dimensional A_1-singularity has infinite order, preventing an existence of a smooth filling. But the A_2-singularity in odd dimension has order 6 (see [15]) and a smooth filling can be found in $\mathcal{C}^\infty$-category. It was remarked long ago by Clemens, Friedman [8, 9] and Morgan that a Calabi–Yau threefold degeneration acquiring an A_2-singularity does not have a smooth filling (see [22]) and there is a result of Voisin [21] implying a same result for four-dimensional varieties acquiring an A_1-singularity.

The purpose of this paper is to present a completely different type of example and analyze it in some detail. The main result is the following.

Theorem. *There exists a flat family* $f : \mathcal{Y} \to \Delta$ *of projective threefolds such that*

(1) $\mathcal{Y}$ *is a smooth fourfold*,
(2) *for* $t \neq 0$, *the fiber* $Y_t := f^{-1}(t)$ *is a smooth Calabi–Yau threefold with*

$$h^{1,1}(Y_t) = 41, \quad h^{1,2}(Y_t) = 1,$$

(3) *the singular locus of* Y_0 *is a line* L, Y_0 *is double along* L *with exactly four pinch points*,
(4) *the elliptic curve* E *doubly covering* L *and ramified over the set* Σ *of these four pinch points has* j*-invariant equal to* 1728,
(5) *the blow-up of* Y_0 *in* L *is a smooth Calabi–Yau threefold* Z_0 *with*

$$h^{1,1} = 46, \quad h^{1,2} = 0,$$

(6) *the local system* $H^i(Y_t)$ *has trivial monodromy over* Δ^* *for* $i \neq 3$ *and* $\mathbb{Z}/2\mathbb{Z}$*-monodromy for* $i = 3$.

Corollary. *The semi-stable reduction, obtained after a base change* $t \mapsto t^2$, *has trivial monodromy and the special fiber consists of two components, one of which is a projective, smooth and rigid Calabi–Yau manifold and the other is a smooth quadric bundle.*

In the Clemens–Friedman–Morgan example we have a Calabi–Yau variety that acquires an A_2-singularity, which is a *terminal singularity.* So from the point of view of the minimal model program, this singular variety should be considered as good as a smooth one. In fact, there does not exist a crepant resolution of the singular member. In our example, the zero-fiber Y_0 has *canonical singularities*, and admits a crepant resolution to an honest smooth Calabi–Yau variety Z_0. There is a change in cohomology in going from Z_0 to Y_t. By Theorem 9.5 of Griffiths ([10]) the period map in fact extends and in Sec. 4 we will see that we can fill in over $t = 0$ the third limiting Hodge structure, which splits as

$$H^3_{\lim}(Y_\infty, \mathbb{Q}) = H^3(Z_0, \mathbb{Q}) \oplus H^1(E, \mathbb{Q})(-1),$$

where E is the elliptic curve mentioned in the theorem. The two pieces of this decomposition are necessarily supported by two different irreducible components of the zero fiber.

The structure of the paper is as follows. In Sec. 1, we describe the basic structure of the example. In Sec. 2, we describe the resolution process in some detail and give the proof of the above theorem. In Sec. 3, we analyze the example on a cohomological level. In Sec. 4, we collect some remarks and speculations.

2. The Example

2.1. *Double octics*

Our example is based on certain special *double octics*, double covers of $\mathbb{P}^3$ ramified along an arrangement of eight planes $P_1, P_2, \ldots, P_8$. Such a space can be given as a hypersurface in a weighted projective space

$$\{u^2 = L_1L_2L_3L_4L_5L_6L_7L_8\} \subset \mathbb{P}[4, 1, 1, 1, 1],$$

where L_i is a linear form defining the plane P_i. For a generic choice of the planes P_i, the branch divisor $D = \cup_{i=1}^{8} P_i$ (and hence also the double cover) has 28 double lines and 56 triple points, along which it is singular. By blowing up the (strict transforms of) the double lines in any order we obtain a crepant resolution which is a Calabi–Yau threefold with Hodge numbers $h^{12} = 9, h^{11} = 29$. For special positions of the planes the singularities of the double octic change, but as long as the configuration of planes does not have fourfold lines or sixfold points, there still exists a crepant resolution, but now the Hodge numbers can take various values, depending on the precise properties of the configuration. Recently, all different cases defining one parameter families of Calabi–Yau threefolds have been listed in [4]. For more information on special double octics we refer to [14].

Our example is based on the following family of double octics

$$X_t := \{u^2 = xy(x + y)z(x + 2y + z + tv)v(y + z + v)(x + y + z + (t - 1)v)\},$$

where $t \in \mathbb{P}^1$ is considered as the parameter. We label the linear forms $L_1, L_2, \ldots, L_8$ in the order as they are written in the above equation:

$$L_1 = x, \quad L_2 = y, \quad L_3 = x + y, \ldots, \quad L_8 = x + y + z + (t - 1)v.$$

For $t \neq 0, 1, 2, \infty$ this configuration has exactly

- a single triple line $\ell_{\text{triple}} : x = y = 0$,
- 25 double lines,
- six fourfold points,

$$(1 : 0 : 0 : 0), \quad (0 : 1 : -1 : 0), \quad (1 : -1 : 1 : 0),$$

$$(t - 2 : 1 : 0 : -1), \quad (1 : 0 : -1 : 0), \quad (1 : -1 : 0 : 0)$$

  not lying on the triple line ℓ_{triple} (called points of *type* p_4^0),

- five fourfold points
$$(0:0:1:0),\quad (0:0:0:1),\quad (0:0:-t:1),$$
$$(0:0:1:-1),\quad (0:0:t-1:-1)$$
on the triple line ℓ_{triple} (called points of *type* p_4^1).

This arrangement is projectively equivalent to arrangement No. 153 in [14] via the coordinate transformation

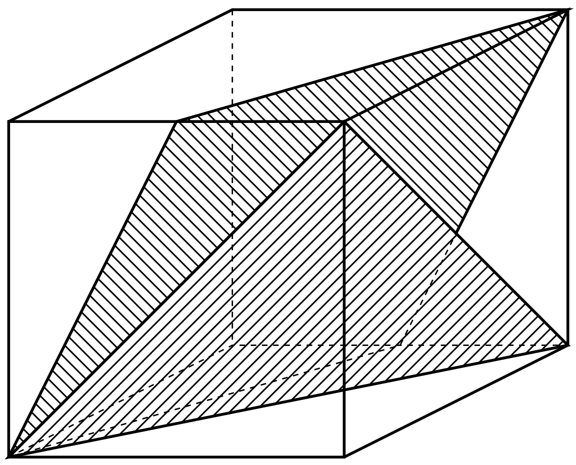
$$(x,y,z,v)\mapsto(-y-z-v,v,y,x+y+z+(t+1)v),$$

and reparametrization $t\mapsto t-2$.

We are here concerned specifically with the degeneration that occurs at $t=0$ and consider the family of double octics over the unit disc:

$$\pi:\mathcal{X}\to\Delta,$$

with fiber over $t\in\Delta$ the double octic X_t defined by the above equation. For this degeneration, there are two important lines, namely the triple line ℓ_{triple} and the moving line $m_t=P_4\cap P_5$. If t goes to 0, the moving line intersects the triple line, and the two fourfold points $\ell_{\mathrm{triple}}\cap P_4$ and $\ell_{\mathrm{triple}}\cap P_5$ move together to form a fivefold point. The result is that for $t=0$ we obtain a configuration equivalent to the rigid arrangement No. 93 of [14].



The plane spanned by the lines ℓ_{triple} and m_0 makes a fourth plane P through ℓ_{triple}. This plane does not belong to the octic arrangement, but the planes P_1,P_2,P_3,P define four points on the projective line L of all planes through ℓ_{triple}. We let E be the double cover of L ramified over these four points. As these planes are in harmonic position, the j-invariant of E is seen to be 1728. This is the elliptic curve we were alluding to in the introduction. We note furthermore that all strata of the singular locus of X_t are defined by the intersections among the planes P_i. As these are not interchanged by the monodromy, these loci form trivial families over Δ^*.

2.2. *Construction of $\mathcal{Y} \to \Delta$*

We will now construct the family $f : \mathcal{Y} \to \Delta$ of the theorem from the family of singular double octics $\mathcal{X} \to \Delta$ by a certain specific sequence of blow-ups. To be more precise, we construct a sequence of blow-ups and a diagram of twofold covers over it:

$$\begin{array}{ccccccc} \mathcal{X}^{(3)} & \longrightarrow & \mathcal{X}^{(2)} & \longrightarrow & \mathcal{X}^{(1)} & \longrightarrow & \mathcal{X} \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ & & \mathbb{P}^{(2)} & \longrightarrow & \mathbb{P}^{(1)} & \longrightarrow & \mathbb{P}^3 \times \Delta. \end{array}$$

We are dealing here with families over Δ and by *blowing-up a line* we mean the blowing-up a *relative* line over Δ, which is really a surface.

- Blow-up $\mathbb{P}^3 \times \Delta$ first in the locus of all fourfold points of type p_4^0 and the triple line. Call the resulting space $\mathbb{P}^{(1)}$ and let $D^{(1)}$ be the strict transform of $D = D^{(0)}$, plus the divisor $E := L \times \ell_{\text{triple}} \times \Delta = \mathbb{P}^1 \times \mathbb{P}^1 \times \Delta$ lying over the triple line $\ell_{\text{triple}} \times \Delta$ over Δ. We denote by $\mathcal{X}^{(1)}$ the double cover ramified over $D^{(1)}$. The branch divisor $D^{(1)}$ contains, apart from the strict transforms of the 25 double lines of D, also eight "new" double lines: three lines m_1, m_2, m_3 in one ruling corresponding to the three planes P_1, P_2, P_3 containing the triple line ℓ_{triple}, and five further lines $m_4, \ldots, m_8$ in the second ruling, namely the intersection of the strict transforms of the planes $P_4, P_5, \ldots, P_8$. In local coordinates near the line m_1 the space $\mathcal{X}^{(1)}$ is described by the equation

  $$u^2 = xyz(x + 2xy + z + t)F.$$

  Here the divisor $x = 0$ is the exceptional set E, $y = 0$ defines the line m_1. The next two factors $z = 0$ and $x + 2xy + z + t = 0$ are the equations of the strict transforms of L_4 respectively, L_5. The factor F is the product of all other factors; it is nonzero on m_1. (We shall consider only the resolution of the complement of lines m_2 and m_3, as the resolution in neighborhoods of these two lines is completely analogous to what happens near m_1).
- In the next step we blow-up all double curves, except three of them: m_4, m_5 and the intersection of strict transforms of planes P_4 and P_5. The new branch divisor $D^{(2)}$ is the strict transform of $D^{(1)}$. Call the resulting double cover $\mathcal{X}^{(2)}$. Over the blow-up of the line m_1 the space $\mathcal{X}^{(2)}$ is described in two charts by

  $$(x, y) \mapsto (xy, y), \quad u^2 = xz(xy + 2xy^2 + z + t)F,$$

  $$(x, y) \mapsto (x, xy), \quad u^2 = yz(x + 2x^2y + z + t)F.$$
- The last step of the resolution is to blow-up the double cover in the union of singular loci of all fibers $\cup_{t\in\Delta} \operatorname{Sing}(X_t)$. We denote by $\mathcal{Y} = \mathcal{X}^{(3)}$ the resulting

variety. To analyze what we have, we look in the two charts of $\mathcal{X}^{(2)}$ around the blow-up of m_1. In the second chart the branch-divisor and the fiber of the branch-divisor are both simple normal crossings. As a consequence, the parts of the spaces $\mathcal{Y}$ and Y_t lying over this chart are smooth. To analyze $\mathcal{Y}$ in the first chart, we have to blow-up $\mathcal{X}^{(2)}$ in the ideal

$$(xz, x(xy + 2xy^2 + z + t), z(xy + 2xy^2 + z + t), u).$$

This blow-up is given as the closure of the map

$$\begin{aligned}(x, y, z, t, u) &\mapsto (X, Y, Z, T)\\ &= (xz, x(xy + 2xy^2 + z + t), z(xy + 2xy^2 + z + t), u).\end{aligned}$$

Using SINGULAR [6] we verified that $\mathcal{Y}$ is smooth in this chart as well and the special fiber is singular along the line

$$\mathrm{Sing}(Y_0) = (x = z = u = X = Y = Z = 0)$$

is contained in the affine chart $T = 1$, moreover in this affine chart the variety Z is locally given as

$$x = XY, \quad z = XZ, \quad u = XYZ, \quad (y + 2y^2)XY + XZ - YZ + t = 0.$$

Proof of the Theorem. *Hodge numbers*: The statement about the Hodge numbers follow from the formulas that express the Hodge-numbers of a resolved double octics X_t and are recorded in [14]: $h^{1,1} = 46, h^{1,2} = 0$ for X_0 and $h^{1,1} = 41$, $h^{1,2} = 1$ for X_t, $t \in \Delta^*$. As the Hodge numbers do not depend on the choice of the resolution, we find the Hodge numbers as stated in the theorem.

Properties of Y_0: The fact that Y_0 is double along the line L and is resolved by a single blow-up follows from the above local calculations. The singular line L of Y_0 can be identified with the pencil of planes through the triple line ℓ_{triple} at the fivefold point $(0 : 0 : 0 : 1)$. Three of the pinch-points correspond to three planes containing ℓ_{triple}, the fourth pinch-point is the direction of the intersection line $P_4 \cap P_5$.

Monodromy: The cohomology group $H^2(Y_t, \mathbb{C}), t \in \Delta^*$ is generated by classes of components of the exceptional locus of the crepant resolution, so it has trivial monodromy. A simple way to determine the monodromy on $H^3(Y_t)$ is using the *Picard–Fuchs operator* $\mathcal{P}$ of the family that was determined in [5]

$$\begin{aligned}\mathcal{P} = 4\Theta\left(\Theta - \frac{1}{2}\right)\left(\Theta - \frac{3}{2}\right)(\Theta - 2) - 12t\left(\Theta - \frac{1}{2}\right)^2\left(\Theta^2 - \Theta + \frac{1}{12}\right)\\ + 13t^2\left(\Theta^4 + \frac{9}{26}\Theta^2 + \frac{1}{208}\right) - 6t^3\left(\Theta + \frac{1}{2}\right)^2\left(\Theta^2 + \Theta + \frac{7}{13}\right)\\ + t^4\left(\Theta + \frac{1}{2}\right)(\Theta + 1)^2\left(\Theta + \frac{3}{2}\right) \in \mathbb{Q}\langle t, \Theta\rangle,\end{aligned}$$

where $\Theta = t\partial/\partial t$ is the logarithmic derivative with respect to the parameter t. The Riemann symbol, that collects the exponents at all singular points is

$$\left\{\begin{array}{cccc} 2 & 1 & 0 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 3/2 & 1 \\ 1 & 1 & 2 & 3/2 \end{array}\right\}.$$

The local system $\mathcal{S}ol$ of solutions on $\mathbb{P}^1\backslash\{0,1,2,\infty\}$ is isomorphic to the local system with fibers $H^3(Y_t,\mathbb{C})$. It can be checked by computing the formal solutions around 0 that *no logarithms occur in the solutions* and one finds four series solutions

$$\phi_0(t) = 1 + \frac{1}{4}t - \frac{23}{1120}t^2 + \cdots, \quad \phi_1(t) = t^{1/2}\left(1 + t + \frac{4}{5}t^2 \dots\right),$$
$$\phi_2(t) = t^{3/2}\left(1 + t + \frac{6}{7}t^2 + \cdots\right), \quad \phi_3(t) = t^2\left(1 + \frac{5}{4}t + \cdots\right).$$

Consequently monodromy for $H^3(Y_t), t \in \Delta^*$ has order 2. □

3. Cohomological Analysis

To describe the cohomological relation between the singular fiber Y_0 and the generic fiber Y_t of our family $f : \mathcal{Y} \to \Delta$, we use the nearby and vanishing cycle formalism from [18]. There is a distinguished triangle in $D^b_{\mathrm{constr}}(Y_0,\mathbb{Q})$ that reads

$$\cdots \to \mathbb{Q}_{Y_0} \to R\Psi_f(\mathbb{Q}) \to R\Phi_f(\mathbb{Q}) \xrightarrow{+1} \cdots$$

and which leads to a long exact sequence in cohomology

$$\cdots \to H^k(Y_0,\mathbb{Q}) \to \mathbb{H}^k(R\Psi_f(\mathbb{Q})) \to \mathbb{H}^k(R\Phi_f(\mathbb{Q})) \to \cdots.$$

The cohomology group $H^k(Y_0,\mathbb{Q})$ carries, after Deligne [7], a natural mixed Hodge structure. The hypercohomology $\mathbb{H}^k(R\Psi_f(\mathbb{Q}))$ can be identified with the cohomology $H^k(Y_\infty,\mathbb{Q})$ of the nearby fiber and carries a canonical mixed Hodge structure after Schmid [19] and Steenbrink [20]. The vanishing cohomology groups

$$\mathbb{H}^k := \mathbb{H}^k(R\Phi_f(\mathbb{Q}))$$

can be given a mixed Hodge structure in a way compatible with this exact sequence.

Proposition. *One has*

$$\mathbb{H}^k = 0 \quad \textit{for } k \neq 3$$

and there is a short exact sequence

$$0 \to H^1(R^2\Phi_f(\mathbb{Q})) \to \mathbb{H}^3 \to H^0(R^3\Phi_f(\mathbb{Q})) \to 0$$

and identifications

$$H^1(R^2\Phi_f(\mathbb{Q})) = H^1(E)(-1),$$
$$H^0(R^3\Phi_f(\mathbb{Q})) = \bigoplus_{p\in\Sigma} \mathbb{Q}(-2)\cdot[p]$$

as MHS.

Proof. We use the hypercohomology spectral sequence $\mathbb{H}^p(R\Phi_f^q(\mathbb{Q})) \Rightarrow \mathbb{H}^{p+q}$. As the singular locus is L, which is codimension 2 in Y_0, we have that $R^0\Phi_f(\mathbb{Q}) = R^1\Phi_f(\mathbb{Q}) = 0$. At a general point $q \in L$ the threefold Y_0 has a transverse A_1-singularity, hence we have that the stalk $R^2\Phi_f(\mathbb{Q})_q$ is one-dimensional, whereas $R^3\Phi_f(\mathbb{Q})_q = 0$. At the pinch-points $p \in \Sigma$ one has $R^2\Phi_f(\mathbb{Q})_p = 0$ and $R^3\Phi_f(\mathbb{Q})_p$ one-dimensional. So $R^3\Phi_f(\mathbb{Q})$ is a sky-scraper sheaf at the pinch points Σ, whereas $R^2\Phi_f(\mathbb{Q})$ is a rank one-local system on $L\backslash\Sigma$, extended to zero. From the local normal form of the pinch-point, one sees that the monodromy on $R^2\Phi_f(\mathbb{Q})$ around the points $p \in \Sigma$ is multiplication by (-1). Hence there is an exact sequence

$$0 \to \mathbb{Q}_L \to n_*(\mathbb{Q}_E) \to R^2\Phi(\mathbb{Q})(1) \to 0,$$

where $n : E \to L$ is the elliptic curve over L, ramified at Σ. From the long exact cohomology sequence we immediately obtain

$$H^0(R^2\Phi_f(\mathbb{Q})) = 0,$$
$$H^1(R^2\Phi_f(\mathbb{Q})) = H^1(E,\mathbb{Q})(-1),$$
$$H^2(R^2\Phi_f(\mathbb{Q})) = 0.$$

The hypercohomology spectral sequences collapse and give the above result. □

Corollary. *The Hodge structures $H^k(Y_0,\mathbb{Q})$ are pure of weight k. There are isomorphisms*

$$H^k(Y_0,\mathbb{Q}) \approx H^k(Y_\infty,\mathbb{Q}) \quad \textit{for } k \neq 3, 4$$

and short exact sequences

$$0 \to H^3(Y_0,\mathbb{Q}) \to H^3(Y_\infty,\mathbb{Q}) \to H^1(E,\mathbb{Q})(-1) \to 0$$
$$0 \to \oplus_{p\in\Sigma}\mathbb{Q}(-2)p \to H^4(Y_0,\mathbb{Q}) \to H^4(Y_\infty,\mathbb{Q}) \to 0.$$

Hence we have

$$h^2(Y_0) = h^2(Y_\infty) = 41, \quad h^3(Y_0) = 2, \quad h^4(Y_0) = h^4(Y_\infty) + 4 = 45.$$

Proof. Note that the Hodge structures $H^k(Y_\infty, \mathbb{Q})$ are pure, as the monodromy is trivial. As only $\mathbb{H}^3 \neq 0$, we get from the long exact cohomology sequence isomorphisms and a five term exact sequence

$$0 \to H^3(Y_0, \mathbb{Q}) \to H^3(Y_\infty, \mathbb{Q}) \to \mathbb{H}^3 \to H^4(Y_0, \mathbb{Q}) \to H^4(Y_\infty, \mathbb{Q}) \to 0.$$

From the fact that $H^3(Y_0, \mathbb{Q}) \neq H^3(Y_\infty, \mathbb{Q})$ and the fact that $H^3(Y_\infty, \mathbb{Q})$ is pure, we see the only possibility is that $H^3(Y_\infty, \mathbb{Q})$ surjects on the weight 3 part $H^1(E, \mathbb{Q})(-1)$ of $\mathbb{H}^3$ and then the kernel of $H^4(Y_0, \mathbb{Q}) \to H^4(Y_\infty, \mathbb{Q})$ is equal to the weight 4 quotient of $\mathbb{H}^3$. □

3.1. *Cohomology of* Z_0

The space Z_0 is obtained from Y_0 by a single blow-up in the line L. The preimage of L is a conic-bundle $Q \to L$. So we get a diagram

$$\begin{array}{ccc} Q & \hookrightarrow & Z_0 \\ \downarrow & & \downarrow \pi \\ L & \hookrightarrow & Y_0. \end{array}$$

The following calculation provides an independent determination of the cohomology of Z_0.

Proposition. *There are exact sequences of mixed Hodge structures*

$$0 \to H^2(Y_0, \mathbb{Q}) \to H^2(Z_0, \mathbb{Q}) \to H^0(R^2\pi_*\mathbb{Q}_{Z_0}) \to 0,$$

$$H^3(Y_0, \mathbb{Q}) \approx H^3(Z_0, \mathbb{Q}),$$

$$0 \to H^4(Y_0, \mathbb{Q}) \to H^4(Z_0, \mathbb{Q}) \to H^2(R^2\pi_*\mathbb{Q}_{Z_0}) \to 0.$$

Furthermore

$$H^0(R^2\pi_*\mathbb{Q}_{Z_0}) = \mathbb{Q}(-1)^5, \quad H^2(R^2\pi_*\mathbb{Q}_{Z_0}) = \mathbb{Q}(-2),$$

hence one gets

$$h^2(Z_0) = h^2(Y_0) + 5 = 46, \quad h^3(Z_0) = h^3(Y_0) = 2, \quad h^4(Z_0) = h^4(Y_0) + 1 = 46.$$

Proof. In order to compare the cohomology of Z_0 and Y_0, we use the Leray spectral sequence

$$E_2^{p,q} = H^p(R^q\pi_*\mathbb{Q}_{Z_0}) \Rightarrow H^{p+q}(Z_0, \mathbb{Q}).$$

The inverse image of L in Y is a conic bundle $Q \to L$. The general fiber is a smooth conic; the fibers over $p \in \Sigma$ are line pairs. Hence

$$R^0\pi_*\mathbb{Q}_{Z_0} = \mathbb{Q}_{Y_0}, \quad R^k\pi_*\mathbb{Q}_{Z_0} = 0, \quad k \neq 0, 2.$$

From the study of the vanishing cohomology for the family $Q \to L$ near the line-pairs, we obtain a split exact sequence

$$0 \to \mathbb{Q}_\Sigma \to R^2\pi_*\mathbb{Q}_Q \to \mathbb{Q}_L \to 0,$$

which indeed leads to

$$H^0(Q) = \mathbb{Q}, \quad H^1(Q,\mathbb{Q}) = 0, \quad H^2(Q,\mathbb{Q}) = \mathbb{Q}(-1)^6,$$
$$H^3(Q,\mathbb{Q}) = 0, \quad H^4(Q,\mathbb{Q}) = \mathbb{Q}(-2).$$

The differential

$$d_3 : H^0(R^2\pi_*\mathbb{Q}_{Z_0}) \to H^3(R^0\pi_*\mathbb{Q}_{Z_0}) = H^3(Y_0,\mathbb{Q})$$

has to be zero because of weights, so we obtain from the spectral sequence short exact sequences as stated above. □

3.2. *The semi-stable reduction*

Denote by $g : \mathcal{Z} \to \mathcal{Y}$ the blow-up of the smooth space $\mathcal{Y}$ in the line L. We denote the exceptional divisor of this blow-up by P; it is a $\mathbb{P}^2$-bundle over L. As the multiplicity of Y_0 alone L is two, the divisor of the composed function

$$h := f \circ g : \mathcal{Z} \to \Delta$$

is

$$Z_0 + 2P,$$

where the strict transform Z_0 of Y_0 is blow-up of Y_0 in L, hence smooth. The intersection of these two components is the surface

$$Q := Z_0 \cap P.$$

The map $Q \to L$, obtained as restriction of g, gives Q the structure of a conic bundle over L; above the four pinch-points the conics degenerate into a line pair. Now we take the pull-back of $h : \mathcal{Z} \to \Delta$ by the squaring map $s : t \mapsto t^2$, and denote its normalization by $\widetilde{\mathcal{Z}}$:

$$\widetilde{\mathcal{Z}} := \widetilde{\Delta \times_\Delta \mathcal{Z}}.$$

We let $n : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ the natural map, so we have the diagram

$$\begin{array}{ccccccccccc}
\widetilde{P} & + & \widetilde{Z_0} & \subset & \widetilde{\mathcal{Z}} & \xrightarrow{n} & \mathcal{Z} & \supset & Z_0 & + & P \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\
\widetilde{L} & \subset & \widetilde{\mathcal{Y}_0} & \subset & \widetilde{\mathcal{Y}} & \longrightarrow & \mathcal{Y} & \supset & Y_0 & \supset & L \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \Delta & \xrightarrow{s} & \Delta. & & & &
\end{array}$$

Proposition. *The space* $\widetilde{\mathcal{Z}}$ *is smooth. The divisor*

$$\widetilde{h}^{-1}(0) = \widetilde{Z_0} + \widetilde{P}$$

is reduced and normal crossing. The map n *induces an isomorphism*

$$\widetilde{Z_0} \to Z_0$$

and a 2*-to-*1 *covering* $\widetilde{P} \to P$ *ramified precisely along* $Q \subset P$.

The cohomology groups $H^i(\widetilde{P}, \mathbb{Q})$ are given by

H^0	H^1	H^2	H^3	H^4	H^5	H^6
$\mathbb{Q}$	0	$\mathbb{Q}(-1)^2$	$H^1(E)(-1)$	$\mathbb{Q}(-2)^2$	0	$\mathbb{Q}(-3)$

Proof. This follows from a direct local calculation. Around any point of $Q = Z_0 \cap P$ the divisor $Z_0 + 2P$ is given by an equation of the form $xy^2 = 0$. The twofold cover then has equation $xy^2 + z^2 = 0$, which has a smooth normalization. Clearly, the map $\widetilde{P} \to P$ is a twofold cover, ramified precisely along the conic bundle Q. In other words, the composition $\rho : \widetilde{P} \to P \to L$ represents this threefold as a quadric bundle, with four fibers with an isolated singular point over the points Σ. We can determine the cohomology of $\widetilde{P}$ using the Leray spectral sequence of the map $\rho : \widetilde{P} \to L$. We find

$$R^0\rho_*(\mathbb{Q}_{\widetilde{P}}) = \mathbb{Q}_L,$$

$$R^1\rho_*(\mathbb{Q}_{\widetilde{P}}) = 0 = R^3\rho_*(\mathbb{Q}_{\widetilde{P}}),$$

$$R^4\rho_*(\mathbb{Q}_{\widetilde{P}}) = \mathbb{Q}(-2).$$

The sheaf $R^2\rho_*(\mathbb{Q}_{\widetilde{P}})$ is more interesting. As H^2 of a quadric is generated by its two rulings, which get interchanged upon surrounding a point of Σ, and coalesce over Σ, we have

$$R^2\rho_*(\mathbb{Q}_{\widetilde{P}}) = \pi_*\mathbb{Q}_E(-1),$$

where $\pi : E \to L$ is the elliptic curve, twofold covering L and ramifying over Σ. The Leray-spectral sequence degenerates and we can read off directly the cohomology groups, as Hodge structures. □

The monodromy weight spectral sequence converges to the cohomology of $H^k(\widetilde{Z}_\infty, \mathbb{Q}) = H^k(Y_\infty, \mathbb{Q})$ and is determined from the intersections of the irreducible components of the semi-stable fiber, see e.g., [17, 20]. In our case there are only two components and a single intersection, so the $E_1^{p,q}$-page is very simple and looks like

$$\begin{array}{lll}
H^4(Q)(-1) \to H^6(Z_0)\oplus H^6(\widetilde{P}) \to & 0, \\
H^3(Q)(-1) \to H^5(Z_0)\oplus H^5(\widetilde{P}) \to & 0, \\
H^2(Q)(-1) \to H^4(Z_0)\oplus H^4(\widetilde{P}) \to & H^4(Q), \\
H^1(Q)(-1) \to H^3(Z_0)\oplus H^3(\widetilde{P}) \to & H^3(Q), \\
H^0(Q)(-1) \to H^2(Z_0)\oplus H^2(\widetilde{P}) \to & H^2(Q), \\
\qquad 0 \qquad\ \to H^1(Z_0)\oplus H^1(\widetilde{P}) \to & H^1(Q), \\
\qquad 0 \qquad\ \to H^0(Z_0)\oplus H^0(\widetilde{P}) \to & H^0(Q).
\end{array}$$

As we have determined all groups appearing, the cohomology-diagram has the following form:

$$\begin{array}{ccccc}
\mathbb{Q}(-3) & \to & \mathbb{Q}(-3)\oplus\mathbb{Q}(-3) & \to & 0, \\
0 & \to & 0\oplus 0 & \to & 0, \\
\mathbb{Q}(-2)^6 & \to & \mathbb{Q}(-2)^{46}\oplus\mathbb{Q}(-2)^2 & \to & \mathbb{Q}(-2), \\
0 & \to & H^3(Z_0,\mathbb{Q})\oplus H^1(E,\mathbb{Q})(-1) & \to & 0, \\
\mathbb{Q}(-1) & \to & \mathbb{Q}(-1)^{46}\oplus\mathbb{Q}(-1)^2 & \to & \mathbb{Q}(-1)^6, \\
0 & \to & 0\oplus 0 & \to & 0, \\
0 & \to & \mathbb{Q}\oplus\mathbb{Q} & \to & \mathbb{Q}.
\end{array}$$

As we know that the cohomology groups $H^k(Y_\infty,\mathbb{Q})$ of the limit are pure Hodge structures, the maps at the left are injective, those on the right surjective, and the cohomology of $H^k(Y_\infty,\mathbb{Q})$ comes out the right way:

$$H^3_{\lim}(Y_0,\mathbb{Q}) = H^3(Z_0,\mathbb{Q})\oplus H^1(E,\mathbb{Q})(-1).$$

4. Outlook

The family $\mathcal{X} \to \Delta$ extends naturally to a projective family over $\mathbb{P}^1$ with four singular fibers at $\{0,1,2,\infty\}$. The fibers at 1 and 2 are double octic arrangements No. 3 and No. 19, respectively, whereas at ∞ we get degenerate configuration

$$u^2 = xy(x+y)zv^3(y+z+v).$$

The map

$$(x,y,z,v,u) \mapsto \left(-x, x+y, z, \frac{1}{t-1}v, u\right)$$

defines an isomorphism between $\mathcal{X}_{\frac{t}{t-1}}$ and the quadratic twist of $\mathcal{X}_t$ by $1-t$, consequently quadratic base-changes of the family $\mathcal{X}$ ramified at 1 and ∞ are isomorphic.

There are 63 one-parameter families of double octics listed in [14], which lead to 63 families of Calabi–Yau threefolds with $h^{12} = 1$. Among these families there are five more examples with a similar behavior. Below one list all six cases, which come as three pairs:

No.	Equation	t_0	$\mathcal{X}_{t_0}$
153	$xyzv(x+y+z)(y+z+v)$ $\times(x-ty+v)(x-ty+z+v)$	-2	93
197	$xyzv(x-y-z+v)(x+ty+tz)$ $\times(ty+tz+v)(x+tz+v)$	$-\frac{1}{2}$	93
96	$xyzv(x+y)(x+y-z+v)$ $\times(x-ty+tz+v)(y+tz+v)$	-2	32
100	$xyzv(x+y-z+v)(x+y+tz)$ $\times(y+tz+v)(ty-tz-v)$	$-\frac{1}{2}$	69
155	$xyzv(x+ty+z)(x+(t+1)y-tz+v)$ $\times(x-tz-tv)(x+ty+z+v)$	$\frac{-1\pm\sqrt{-3}}{2}$	A
200	$xyzv(x+y+z+v)(x+y-tz-tv)$ $\times(y-tz+v)(x-ty-tv)$	$\frac{-1\pm\sqrt{-3}}{2}$	A

In the last column we have indicated the configuration number of the corresponding double octic from [14]. The symbol A indicates a specific rigid Calabi–Yau manifold defined over $\mathbb{Q}(\sqrt{-3})$. The families No. 96 and No. 100 are in fact birational, as are No. 153 and No. 197. Families No. 155 and No. 200 have equal Hodge numbers and share the same Picard–Fuchs operators, but no birational map between them is known to us.

The degeneration of two fourfold points of type p_4^1 that collide and produce a p_5^1 that was analyzed in this paper for No. 153 also occurs in No. 100 and No. 155. As a consequence we get again the central fiber singular along a double line with four pinch points. The only difference is that in the case of family No. 155 the j-invariant of four pinch-points equals 0. The degenerations that occur in the other three cases No. 96, No. 197 and No. 200 are of a different kind: three double lines come together to form a triple line. This line is a double line of the singular element of the central fiber with four pinch point: one fivefold point and three fourfold points on this line. The j-invariant of this four points is again 1728 in first two cases and 0 in the last case.

The local exponents of the Picard–Fuchs operators in the first four families are all equal to $(0, 1/2, 3/2, 2)$, which after quadratic base change become $(0, 1, 3, 4)$, while in the case of the last two families No. $155, 200$ they are $(0, 1/2, 5/2, 3)$, which after a quadratic base change become $(0, 1, 5, 6)$. It is surprising and beautiful to see the order of the automorphism group of the associated elliptic curve appear in the local exponents of the degeneration.

The first four families No. $96, 100, 153$ and 197 are also birational to Kummer fibrations of rational elliptic surfaces. The degeneration of the corresponding fiber products results from the collisions of fibers

$$(I_2 \times I_0^*) + (I_0 \times I_0) \to (I_2 \times I_0^*) \quad \text{or} \quad (I_2 \times I_0^*) + (I_2 \times I_0) \to (I_4 \times I_0^*).$$

The singularities in the central fiber correspond to two copies of the singular fiber of type I_0^*. It may very well be possible to analyze the degeneration cohomologically from this description.

We believe that our degeneration also has an arithmetical version that may be of interest. Recently, in [3, 13] a version Néron–Ogg–Shafarevich criterion for a family of K3 surfaces was formulated, which enables to detect good reduction of a K3 surface over the fraction field K of a henselian local ring R with residue field k of characteristic $p > 0$ by having the Galois representation

$$G_K \to \mathrm{Aut}(H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell))$$

unramified. Based on our example, we are inclined to think that no similar criterion can exist for Calabi–Yau threefolds. If we replace t by a sufficiently large prime p in the formula describing our double octic, and doing the corresponding modifications, we end up with a Calabi–Yau variety Y over the p-adic field $K = \mathbb{Q}_p$ for which we have the suspicion that the Galois representation

$$G_K \to \mathrm{Aut}(H^3_{\mathrm{et}}(Y, \mathbb{Q}_\ell))$$

is unramified for $\ell \neq p$ and crystalline for $\ell = p$, but for which no good (terminal) reduction is in sight.

Acknowledgments

We thank Radu Laza for showing interest in this example. The first named author was partially supported by the National Science Center Grant no. 2014/13/B/ST1/00133. This work was partially supported by the Grant 346300 for IMPAN from the Simons Foundation and the matching 2015–2019 Polish MNiSW fund.

References

[1] M. Atiyah, On analytic surfaces with double points, *Proc. Roy. Soc. A* **247** (1958) 237–244.

[2] E. Brieskorn, Singular elements of semi-simple algebraic groups, *Actes. Congrès Inter. Math.* **2** (1970) 279–284.

[3] B. Chiarellotto, C. Lazda and C. Liedtke, A Néron–Ogg–Shafarevich criterion for K3-surfaces, *Proc. Lond. Math. Soc.* (*3*) **119** (2019) 469–514.

[4] S. Cynk and B. Kocel-Cynk, Classification of double octic Calabi–Yau threefolds, *Commun. Contemp. Math.* **22** (2020); Article ID:1850082, 38 pp.

[5] S. Cynk and D. van Straten, Picard–Fuchs operators for octic arrangements I (The case of or phans), *Commun. Number Theory Phys.* **13** (2019) 1–52.
[6] W. Decker, G. M. Greuel, G.-Pfister and H. Schönemann, Singular 4-1-1 — A computer algebra system for polynomial computations; http://www.singular.uni-kl.de (2018).
[7] P. Deligne, *Théorie de Hodge. I.* Actes du Congrés International des Mathématiciens, Tome 1 (Gauthier-Villars, Paris, 1971), pp. 425–430.
[8] R. Friedman, A degenerating family of quintic hypersurfaces with trivial monodromy, *Duke Math. J.* **50** (1983) 203–214.
[9] R. Friedman, On threefolds with trivial canonical bundle, in *Complex Geometry and Lie Theory*, editions J. Carlson, H. Clemens and D. Morrison, Proceedings of Symposia in Pure Mathematics, Vol. 53 (American Mathematical Society, 1991), pp. 103–134.
[10] P. Griffiths, Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping, *Inst. Hautes Études Sci. Publ. Math.* **38** (1970) 125–180.
[11] J. Kollár, R. Laza, G. Sacca and C. Voisin, Remarks on degenerations of Hyper-Kähler manifolds, *Ann. Inst. Fourier* (*Grenoble*) **68**(7) (2018) 2837–2882.
[12] V. S. Kulikov, Degenerations of K3 surfaces and enriques surfaces, *Math. USSR Izv.* **11** (1977) 957–989.
[13] C. Liedtke and Y. Matsumoto, Good reduction of K3 surfaces, *Compos. Math.* **154** (2018) 1–35.
[14] C. Meyer, *Modular Calabi–Yau Threefolds*, Fields Institute Monographs, Vol. 22 (American Mathematical Society, Providence, RI, 2005).
[15] J. Morgan, Topological triviality of various analytic families, *Duke Math. J.* **50** (1983) 215–225.
[16] U. Persson and H. Pinkham, Degeneration of surfaces with trivial canonical bundle, *Ann. of Math.* **113** (1981) 45–66.
[17] C. Peters and J. Steenbrink, *Mixed Hodge Structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Vol. 52 (Springer-Verlag, Berlin, 2008).
[18] P. Deligne and N. Katz, Groupes de monodromie en géométrie algébrique. II. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Lecture Notes in Mathematics, Vol. 340 (Springer-Verlag, Berlin, 1973).
[19] W. Schmid, Variation of Hodge structure: The singularities of the period mapping, *Invent. Math.* **22** (1973) 211–319.
[20] J. Steenbrink, Limits of Hodge structures, *Invent. Math.* **31** (1975/76) 229–257.
[21] C. Voisin, Degenerations de Lefschetz et variations de structures de Hodge, *J. Differential Geometry* **31** (1990) 527–534.
[22] C.-L. Wang, On the incompleteness of the Weil-Petersson metric along degenerations of Calabi–Yau manifolds, *Math. Res. Lett.* **4** (1997) 157–171.